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Radially Symmetric Zero-Potential Densities for the Solution of Gravitational Inverse Problems

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Radially Symmetric Zero-Potential Densities for the Solution of Gravitational Inverse Problems

Sylvia Krappmann, Helmut Moritz, Graz

Abstract

There are infinitely many density distributions that are compatible with a given external potential *(gravitational inverse problem).* This non-uniqueness is best expressed by the possibility to add arbitrary *zero-potential densities* which change the mass distribution without affecting the external potential. The present paper investigates special *radially symmetric* zero-potential densities, which can be used as some kind of *spline functions* for approximating more general zero-potential densities.

Zusammenfassung

Ein gegebenes Außenraumpotential kann durch unendlich viele Dichteverteilungen erzeugt werden *(inverses gravimetrisches Problem).* Diese Mehrdeudigkeit wird am besten dadurch ausgedrückt, daß man beliebige *Nullpotentialdichten* addieren kann, die zwar die Massenverteilung verändern, aber keine Wirkung auf das Außenraumpotential ausüben. Im vorliegenden Artikel werden spezielle *radialsymmetrischen* Nullpotentialdichtenuntersucht, die für die Approximation allgemeinerer Nullpotentialdichten als eine Art *Spline-Funktionen* verwendet werden können.

1. Introduction

The gravitational potential V of a body is given by the well-known Newton integral

$$V(P) = G \iiint_v \frac{\rho(Q)}{l_{PQ}} dv_Q, \qquad (1)$$

where P denotes the point at which V is to be computed and Q is the variable integration point to which the density ρ and the volume element v refer; l_{PQ} is the distance between P and Q; the gravitational constant is denoted by G as usual.

It is an essential point that the potential V is *linear* in the density ρ, cf. [4]. Thus we can write

$$V = N\,\rho, \qquad (2)$$

where N defined by the integral (1) denotes the linear *Newtonian operator.* If the external potential V is given, we solve the *gravitational inverse problem* by inverting (2)

$$\rho = N^{-1}\,V, \qquad (3)$$

which provides us an opportunity to determine the unknown density ρ. But the operator N^{-1} is a nonunique quantity and this is the reason why (3) has infinitely many solutions for the density ρ.

The general solution of the inhomogeneous equation (2) is obtained as the sum of the *uniquely defined harmonic density* ρ_H, which satisfy Laplace's equation $\Delta\rho_H = 0$, and *the set of zero-potential densities* ρ_0, cf. [4],

$$\rho = \rho_H + \rho_0. \qquad (4)$$

The set of zero-potential densities ρ_0 comprises all density distributions within the surface S which produces zero external potential

$$N\,\rho_0 = V_0 = 0. \qquad (5)$$

The result of (5) is nonunique. Thus the ambiguity of the general solution (4) is expressed by the set of possible zero-potential densities ρ_0. The condition of zero external potential is the absence of gravitational attraction outside the body and this will only be satisfied if the *total mass of the body is zero.* Therefore the densities ρ_0 must be alternatively positive and negative. But physically there are no negative densities, so the "densities" ρ_0 represent *density anomalies* and the potentials V_0 are the corresponding *potential anomalies.*

In the following sections we shall examine special *radially symmetric* zero-potential densities. We suppose that the boundary surface S of the body is a sphere. In many cases the sphere represents a sufficient approximation to the earth as far as *potential anomalies* are concerned.

2. Radially Symmetric Zero-Potential Densities

As a preparation for the approximation of more general zero-potential densities, let us expand a *radially symmetric* zero-potential density function

$\rho_0(\bar{r})$ into a normalized (division of $\bar{r}$ by R) polynomial which is restricted to even powers, cf. [2],

$$\rho_0(\bar{r}) = \sum_{k=0}^{N} a_k \left(\frac{\bar{r}}{R}\right)^{2k}, \qquad (6)$$

where a_k denote constant coefficients. Now the following question arises: What condition must the constant coefficients a_k satisfy to produce zero-potential densities ρ_0? The condition for zero-potential densities is that the *total mass* of the body becomes zero. Therefore the linear dependence of the constant coefficients (a_0, a_1,..., a_N) can be derived as follows.

The total mass M of a sphere, cf. [4], which has to be zero in this case, can be determined by

$$M = 4\pi \int_{\bar{r}=0}^{R} \rho_0(\bar{r})\bar{r}^2 \, d\bar{r} = 0. \qquad (7)$$

Replacing $\rho_0(\bar{r})$ by the definition (6), the solution of (7) gives the general condition of the constant coefficients (a_0, a_1,..., a_N) of ρ_0:

$$\sum_{k=0}^{N} \frac{a_k}{2k+3} = 0, \qquad (8)$$

to be satisfied by zero-potential densities (6).

To determine an explicit solution for zero-potential densities, we extend the polynomial (6) to sixth-order, which means $N = 3$. Setting $a_0 = a$, $a_1 = b$, $a_2 = c$ and $a_3 = d$, we obtain for the unit sphere ($R = 1$) by (6)

$$\rho_0(\bar{r}) = a + b\bar{r}^2 + c\bar{r}^4 + d\bar{r}^6. \qquad (9)$$

As we have seen above, the constant coefficients (a, b, c, d) must satisfy the necessary condition of zero-potential density (8). Besides we require that the function (9), together with its first derivative, must be zero at the boundary. In addition we normalize $a = 1$. Taking all these conditions into account, we finally get *one* of the infinitely many zero-potential density distributions as

$$\rho_0(\bar{r}) = 1 - 5\bar{r}^2 + 7\bar{r}^4 - 3\bar{r}^6. \qquad (10)$$

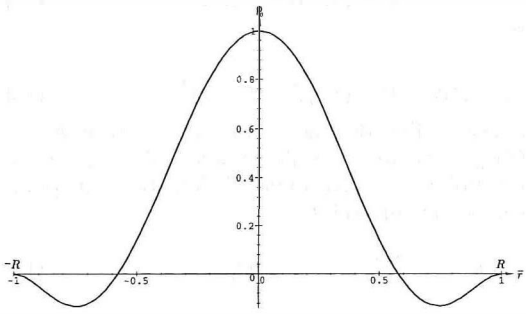

Fig. 1: A zero-potential density function $\rho_0(\bar{r})$ for one dimension

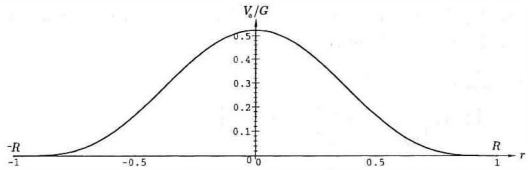

Fig. 2: The zero-potential distribution $V_0(\bar{r})/G$ corresponding to Fig.1

Figure 1 shows the zero-potential density distribution (10), for one dimension. As we can see in this figure, the function $\rho_0(\bar{r})$ has positive and negative values and is continuous and differentiable everywhere (also on the surface S). The next figure, Fig. 2, illustrates the corresponding zero-potential distribution (divided by the gravitational constant G). The potential inside the sphere is determined by the well-known Lauricella's second theorem, cf. [4] and [2].

3. Zero-Potential Splines

Now we try to approximate arbitrary zero-potential density distributions ρ_0 by a linear combination of *zero-potential splines,* in analogy to the approximation of potentials by discrete point masses.

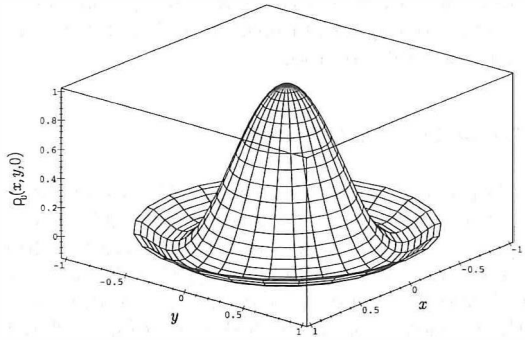

Fig.3: The spline function $s(\bar{r}_i, R)$ for two dimensions: $R = 1$

Zero-potential splines are defined in the following way, cf. [2]. Assume that there are N spheres of the same radius R = *const.*, of centres M_i ($1 \leq i \leq N$), and of radially symmetric zero-potential density distributions $\rho_0(\bar{r}_i, R)$ according to (10), see also Fig. 3,

$$\rho_0(\bar{r}_i, R) = 1 - 5\left(\frac{\bar{r}_i}{R}\right)^2 + 7\left(\frac{\bar{r}_i}{R}\right)^4 - 3\left(\frac{\bar{r}_i}{R}\right)^6 \qquad (11)$$

where $\bar{r}_i$ *is the distance from the centre* M_i to point Q at which the density should be determined. But also any multiple of this function gives a possible solution. Thus we introduce a

"weight factor" α_i which can be prescribed rather arbitrarily and gives us

$$\rho_{0,i}(\bar{r}_i, R) = \alpha_i\, s(\bar{r}_i, R) = \\ = \alpha_i \left[1-5\left(\frac{\bar{r}_i}{R}\right)^2+7\left(\frac{\bar{r}_i}{R}\right)^4-3\left(\frac{\bar{r}_i}{R}\right)^6\right]. \quad (12)$$

The function $s(\bar{r}_i, R)$ is called *zero-potential spline,* since it has the character of a spline function in the following sense. It is a *finite function* which means that its support is finite. The support of a function is the region in which the function is different from zero. Obviously, in our case the support of the spline function is a sphere of radius R.

Now we want to know the density ρ_0 at an arbitrary point Q. For this purpose we approximate the unknown density $\rho_0\,(Q)$ by a linear combination of the given splines:

$$\rho_0\,(Q) = \sum_{i=1}^{N} \alpha_i\, s(\bar{r}_i, R). \quad (13)$$

The summation takes all N splines $s(\bar{r}_i, R)$ into account, but note: If $\bar{r}_i > R$), the value of the spline is zero, which means no effect of the corresponding spline exists. Only if $0 \le \bar{r}_i \le R$) is satisfied, the corresponding spline is used for the approximation of the unknown density ρ_0 at the special point Q, but this depends on the geometric situation. In the following sections different simple models will be investigated in three-dimensional space, in order to get a first understanding of the method.

4. Models with 27 Spheres

Let us put N spheres (centres M_{ijk}, radius R) in a cube of length L. Place the centres M_{ijk} of the spheres at the nodes of a cubical grid inside the cube. The mesh width of the grid is denoted by h, being the same in each direction of the axes. The configuration of the nodes M_{ijk} of the cubical grid should be symmetric with respect to the origin O (centre of the cube (L)). Symmetric with respect to the origin O means that each point M_{ijk} has an opposite point in the direction of the origin O, which is the centre of the distance between such a pair of points. Therefore the centre of the cubical grid must be the origin O. Further the outer spheres should touch the boundary of the cube (L), see also Figure 4 and 6.

So the basic configuration has been fixed, but we can still choose the number of the spheres $N = n^3$ (n = positive integer), the radius R which is the same for all spheres, and the mesh width h of the centres M_{ijk} of the spheres. Since the outer spheres should touch the boundary of the cube, the length L of the cube can be determined by

$$L = 2R + nh. \quad (14)$$

Remark: In this paper the name of a vector (small letter) and of a matrix (capital letter) is written in boldface, and a three-dimensional tensor is represented by an *underlined* boldface letter.

Now let us consider an example for $n = 3$. In this case we get $N = 3 \times 3 \times 3 = 27$ spheres of constant radii R. The distance h can be prescribed arbitrarily. Different distances h give different lengths L of the cube (14).

To get more symmetry, let the indices i (x-direction), j (y-direction), and k (z-direction) run from –1 to 1. Consider the following tensor $\underline{\mathbf{M}}$ with ist centre-elements M_{ijk}

$$\underline{\mathbf{M}} = \left[\begin{bmatrix} M_{-1,-1,1} & M_{-1,0,1} & M_{-1,1,1} \\ M_{0,-1,1} & M_{0,0,1} & M_{0,1,1} \\ M_{1,-1,1} & M_{1,0,1} & M_{1,1,1} \end{bmatrix} \begin{bmatrix} M_{-1,-1,0} & M_{-1,0,0} & M_{-1,1,0} \\ M_{0,-1,0} & M_{0,0,0} & M_{0,1,0} \\ M_{1,-1,0} & M_{1,0,0} & M_{1,1,0} \end{bmatrix} \begin{bmatrix} M_{-1,-1,-1} & M_{-1,0,-1} & M_{-1,1,-1} \\ M_{0,-1,-1} & M_{0,0,-1} & M_{0,1,-1} \\ M_{1,-1,-1} & M_{1,0,-1} & M_{1,1,-1} \end{bmatrix}\right] \quad (15)$$

Comparing (15) with Fig. 4, we see that the tensor reflects the geometrical situation of the centres M_{ijk}. Index k is equivalent to the number of the corresponding plane.

Now, at any centre there is a corresponding spline function $s(\bar{r}_{ijk}, R)$. Since R is constant, we can write

$$s(\bar{r}_{ijk}, R) = s(\bar{r}_{ijk}) = s(M_{ijk}, Q) = s_{ijk}(Q), \quad (16)$$

which is defined by analogy to (11) as

$$s_{ijk}\,(Q) = \begin{cases} 1-5\left(\dfrac{\bar{r}_{ijk}}{R}\right)^2+7\left(\dfrac{\bar{r}_{ijk}}{R}\right)^4-3\left(\dfrac{\bar{r}_{ijk}}{R}\right)^6 \\ \\ 0 \text{ on and outside } S \end{cases} \quad (17)$$

where

$$\bar{r}_{ijk} = \sqrt{(x-\bar{x}_i)^2 + (y-\bar{y}_j)^2 + (z-\bar{z}_k)^2} \quad (18)$$

denotes the distance from the centre $M_{ijk} = M(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ to an arbitrary point $Q(x, y, z)$. Then the unknown zero-potential density ρ_0 at point $Q(x, y, z)$ is given by

$$\rho_0(Q) = \sum_{k=-1}^{1} \sum_{j=-1}^{1} \sum_{i=-1}^{1} \alpha_{ijk}\, s_{ijk}(Q), \quad (19)$$

a superposition of spline functions s_{ijk} with different scales α_{ijk}, according to (13).

Originally we have 27 α_{ijk}, which can be chosen arbitrarily. On the other hand they can be determined by prescribing definite values of ρ_0 at data points Q_{ijk}. We thus assume our zero-potential densities to be the result of the superposition of splines given at special points Q_{ijk}. Then the following question arises: Which coefficients α_{ijk} produce the given values at Q_{ijk}?

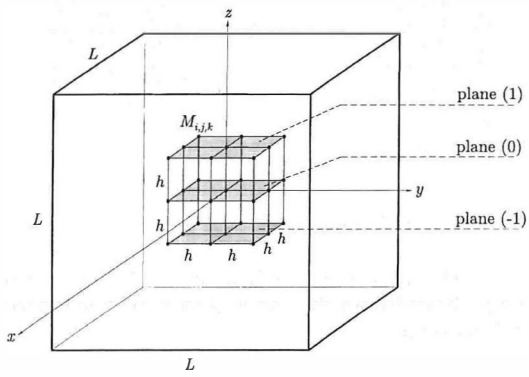

Fig. 4: Positions of the centres M_{ijk} of the spheres (R) inside the cube (L)

Generally we need values at 27 points Q_{ijk} to determine the 27 unknown α_{ijk}. Let these values be summarized in vectors **l** and **a**

$$\mathbf{l} = \begin{pmatrix} \rho_0(Q_{-1,-1,-1}) \\ \rho_0(Q_{-1,-1,0}) \\ \vdots \\ \rho_0(Q_{1,1,1}) \end{pmatrix} \qquad \mathbf{a} = \begin{pmatrix} \alpha_{-1,-1,-1} \\ \alpha_{-1,-1,0} \\ \vdots \\ \alpha_{1,1,1} \end{pmatrix}.$$

Note that the indices of the points Q_{ijk} do not refer to the positions of these points, since Q_{ijk} can lie almost everywhere inside the cube (L), but the indices refer to the centres M_{ijk} of the splines in the sense that we need a value ρ_0 at a point Q_{ijk} for each unknown α_{ijk} of the corresponding sphere centered at M_{ijk}. According to (19), we obtain

$$\mathbf{l}^T = \mathbf{a}^T \mathbf{B}, \tag{20}$$

where the matrix **B** (27×27) consists of the spline elements s_{ijk} (Q_{ijk}). Note that the coefficient vector **a** is constant whereas the spline function matrix **B** depends on the variable points $Q_{ijk}(x, y, z)$, see (17) and (18). Thus the coefficients α_{ijk} can be determined by inversion of matrix **B**

$$\mathbf{a}^T = \mathbf{l}^T \mathbf{B}^{-1}. \tag{21}$$

In order to understand a method, it is frequently useful to *apply it to a very simple but very extreme case.* Therefore let us introduce some simplifications. Firstly, assume that the result of the superposition is symmetric with respect to the origin O, and the data points Q_{ijk} should correspond to nodes of a cubical grid in the middle of the cube (L). The mesh width is denoted by $\bar{h}$ ($0 < \bar{h} < L/2$) being the same in each direction of the axes.

Considering the 27-point cube of length $2\bar{h}$, we notice that the points can be divided into 4 kinds (0,...,3) on the basis of the 4 different kinds of diagonals regarding the origin O (Fig. 5). Each point of *one* kind has the same distance to the origin O whereas points of *different* kinds belong to different kinds of diagonals regarding the origin O. Taking *one* point of *each* kind, i.g.

$$\tilde{Q}_0 = Q_{0,0,0}(0,0,0) \quad \tilde{Q}_2 = Q_{0,1,1}(0, \bar{h}, \bar{h})$$
$$\tilde{Q}_1 = Q_{0,0,1}(0,0,\bar{h}) \quad \tilde{Q}_3 = Q_{1,1,1}(\bar{h}, \bar{h}, \bar{h}) \tag{22}$$

we obtain 4 points $\tilde{Q}_m \in Q_{ijk}$ ($m = 0,...,3$).

Secondly, we assume that *the values at the points of one kind $\tilde{Q}_m$ are the same.* Then we obtain a result of zero-potential density distribution, which is completely symmetric with respect to the origin O. But further this means that also the data α_{ijk} have to be symmetric with respect to the origin O.

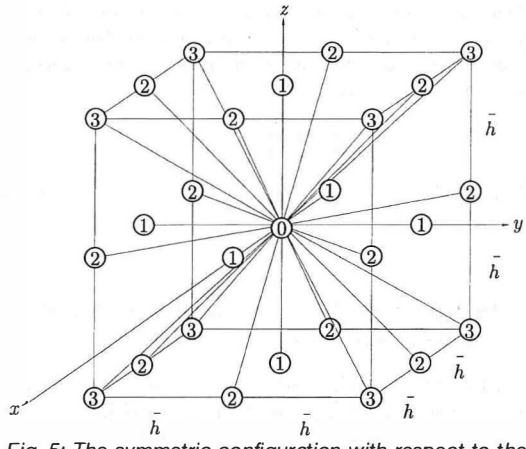

Fig. 5: The symmetric configuration with respect to the origin O

In this sense we have only 4 different unknown coefficients ($\tilde{\alpha}_0$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$) according to Fig. 5. The coefficient $\tilde{\alpha}_0$ occurs once, $\tilde{\alpha}_1$ 6 times, $\tilde{\alpha}_2$ 12 times and $\tilde{\alpha}_3$ 8 times. For better understanding see the following symmetric tensor $\underline{\mathbf{A}}$:

$$\underline{\mathbf{A}} = [\alpha_{ijk}] = \left[\begin{bmatrix} \tilde{\alpha}_3 & \tilde{\alpha}_2 & \tilde{\alpha}_3 \\ \tilde{\alpha}_2 & \tilde{\alpha}_1 & \tilde{\alpha}_2 \\ \tilde{\alpha}_3 & \tilde{\alpha}_2 & \tilde{\alpha}_3 \end{bmatrix} \begin{bmatrix} \tilde{\alpha}_2 & \tilde{\alpha}_1 & \tilde{\alpha}_2 \\ \tilde{\alpha}_1 & \tilde{\alpha}_0 & \tilde{\alpha}_1 \\ \tilde{\alpha}_2 & \tilde{\alpha}_1 & \tilde{\alpha}_2 \end{bmatrix} \begin{bmatrix} \tilde{\alpha}_3 & \tilde{\alpha}_2 & \tilde{\alpha}_3 \\ \tilde{\alpha}_2 & \tilde{\alpha}_1 & \tilde{\alpha}_2 \\ \tilde{\alpha}_3 & \tilde{\alpha}_2 & \tilde{\alpha}_3 \end{bmatrix} \right] . \quad (23)$$

Taking the symmetric situation into account, we also need only values summarized in vector $\tilde{\mathbf{l}}$ at *one* point of *each* kind to determine these 4 different coefficients of vector $\tilde{\mathbf{a}}$

$$\tilde{\mathbf{l}} = \begin{pmatrix} \rho_0(\tilde{Q}_0) \\ \rho_0(\tilde{Q}_1) \\ \rho_0(\tilde{Q}_2) \\ \rho_0(\tilde{Q}_3) \end{pmatrix} \in \mathbf{l} \qquad \tilde{\mathbf{a}} = \begin{pmatrix} \tilde{\alpha}_0 \\ \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \\ \tilde{\alpha}_3 \end{pmatrix} \in \mathbf{a}.$$

By analogy to (20) we get

$$\tilde{\mathbf{l}}^T = \tilde{\mathbf{a}}^T \tilde{\mathbf{B}}. \quad (24)$$

Thus $\tilde{\mathbf{a}}$ is obtained by inversion of $\tilde{\mathbf{B}}$

$$\tilde{\mathbf{a}}^T = \tilde{\mathbf{l}}^T \tilde{\mathbf{B}}^{-1} \quad . (25)$$

and the 4 coefficients ($\tilde{\alpha}_0$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$) of $\tilde{\mathbf{a}}$ can be substituted into (23).

At the next stage we put the above assumptions in concrete form. The 27 points Q_{ijk} should correspond to the 27 centres M_{ijk} of the spline functions and *all values at these points should be the same,* namely 1:

$$Q_{ijk} = Mijk \qquad \mathbf{l} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} . \quad (26)$$

Thus we have the same situation as described above only with $\tilde{h} = h$.

Models with different mesh widths h are investigated in [2]. Using small mesh widths, a very surprising result appears which will be described now.

For the following concrete calculation we introduce the unit sphere (R = 1). In order to illustrate the zero-potential density distribution inside the cube (L) graphically we cut the cube by the plane z = 0. The point Q varies all over this plane inside the cube. Let the mesh width h be $R/4$. The geometrical situation is seen in Fig. 6.

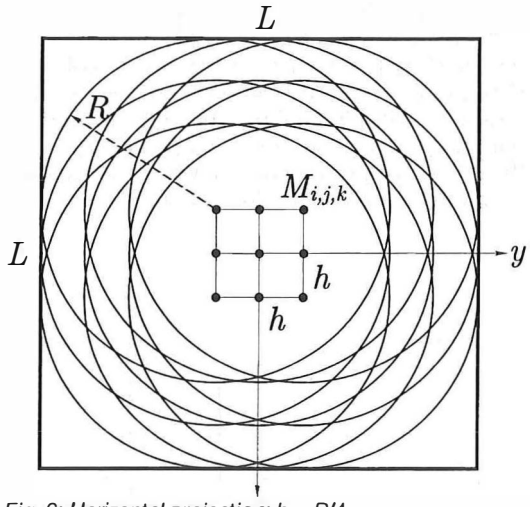

Fig. 6: Horizontal projection: h = R/4

We determine the coefficients α_{ijk} by inversion (25). Substituting the result into (23), our tensor $\underline{\mathbf{A}}$ becomes

$$\underline{\mathbf{A}} = [\alpha_{ijk}] = -\frac{1}{144\, h^6} \left[\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 & -2 \\ 4 & -8 & 4 \\ -2 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \right] . \quad (27)$$

In Fig. 7 we see the superposition of the corresponding zero-potential splines.

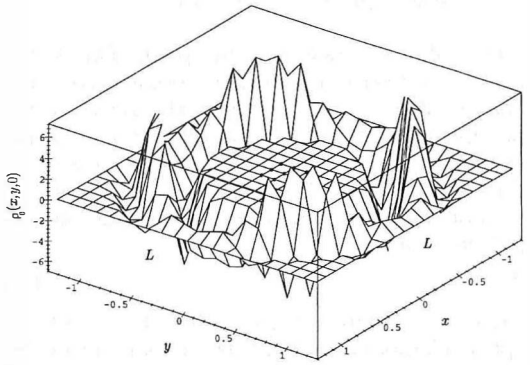

Fig. 7: Superposition (h = R/4)

Now two interesting questions arise in an almost miraculous way:

Miracle 1: Why is the result in the middle area of Fig. 7 a large *plane* at a height of 1?

Miracle 2: Why are the coefficients ($\tilde{\alpha}_0$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$) of (27) multiples (–8, 4, –2, 1) of the factor $1/(-144h^6)$?

The configuration of the centres M_{ijk} in Fig. 6 has an special property: All centres M_{ijk} lie in a region (intersection of volume), called *"critical region"*, where all 27 spline functions s_{ijk}, restricted to their corresponding supports (spheres), are superimposed that is, they are polynomials $\not\equiv 0$. *The plane at a height of 1 is bounded by this "critical region".* (The "critical region" is the plateau in the central part of Fig. 7.) This special configuration of the centres M_{ijk} is always satisfied if the space diagonal of the grid (M_{ijk}) is not greater than the radius R of the spheres, which gives the condition $0 \leq h \leq R/2\sqrt{3}$. Whenever h lies in this region, the two miracles described above occur.

It is miraculous indeed that the superposition of zero-potential splines, which are polynomials of higher order, should give an exact plane, which represents a linear function. That the plane is really exact, can be shown by direct computation using (19). Equation (19) can also be written as

$$\rho_0(Q) = \mathbf{a}^T \mathbf{b} = \underline{\mathbf{A}} \odot \underline{\mathbf{B}} \equiv 1, \tag{28}$$

where the tensor $\underline{\mathbf{B}}$ reflects the geometric situation of its element s_{ijk} determined by substituting (18) into (17), cf. also (15). The sign $\odot$ in (28) is defined as the summation over all products of matrices (tensors) – elements with equivalent indices.

Note once more the result is the constant value 1 *identically in the whole "critical region"* defined above, not only at the given data points Q_{ijk}!

The fact that the zero-potential density function corresponds to a plane throughout the "critical region" ("Miracle 1"), is so surprising that one is very eager to find an explanation. Perhaps Miracle 1 can be explained by Miracle 2?

We know that the discretization of a differential operator lead to matrices which are quite similar to the matrices occurring in (27). This will be considered now.

5. Difference Operators for Approximation of Differential Operators

The approximation of partial derivatives by finite differences is discussed in [1]. As a matter of fact, the derivative of a function with respect to x, y, or z can be approximated by a difference of values of this function at discrete points. In this section *only one spline function of a unit sphere* ($R = 1$) whose centre corresponds to the origin O of the coordinate-axes is considered and can be denoted by $s_0(x, x, z)$ since

$$s(\bar{r}) = s(M, Q) = s(O, Q) = s_{0,0,0}(Q) = s_0(Q) = s_0(x, y, z). \tag{29}$$

Substituting $\bar{r}$ which is the distance from the centre $O(0,0,0)$ to the variable point $Q(x, y, z)$

$$\bar{r} = \sqrt{x^2 + y^2 + z^2} \tag{30}$$

into (17) we obtain the spline function depending on the position (x, y, z) of point Q.

Let us consider only the case if point Q lies inside the range $0 \leq \bar{r} \leq R$. Assume that we introduce a symmetric cubical grid around point Q so that point $Q(x, y, z)$ corresponds to the central node $D_{0,0,0}(x, y, z)$ of the grid. The point Q is not necessarily equal to the origin O! The mesh width denoted by h is the same in each direction of the axes. Now the 27 nodes of the grid D_{ijk} (1 central node, 26 outer nodes) should be the discrete points for the approximation. Our aim is to use central differences (differences between outer nodes and the central node) to approximate derivatives of the spline function $s_0(x, y, z)$ at the central node $D_{0,0,0}(x, y, z) = Q(x, y, z)$.

Now let us regard the mixed square derivative of a single spline function $s_0(x, y, z)$ (29) which gives

$$\left(\frac{\partial^6 s_0}{\partial x^2\, \partial y^2\, \partial z^2}\right)_Q = -144, \tag{31}$$

identically for *all* points (x, y, z) in the region $0 \leq \bar{r} \leq R$. At the next stage we replace the mixed square derivative (31) by a difference operator with $0 < h \leq R/2\sqrt{3}$. Using the approximate representation by finite differences, cf. [1], we obtain

$$\left(\frac{\partial^6 s_0}{\partial x^2\, \partial y^2\, \partial z^2}\right)_Q \cong \left(\frac{\partial^6 s_0}{\partial x^2\, \partial y^2\, \partial z^2}\right)^*_Q = \frac{1}{h^6}\,(\underline{\mathbf{A}}^* \odot \underline{\mathbf{B}}^*) = \frac{1}{h^6}\,(-144h^6) = -144, \tag{32}$$

where the difference operator $\underline{\mathbf{A}}^*$ is equal to $-144h^6\,\underline{\mathbf{A}}$ of (27) and tensor $\underline{\mathbf{B}}^*$ reflects the geometric situation of the elements $s_0(D_{ijk})$. By direct computation we see that tensor $\underline{\mathbf{B}}^*$ is identical to $\underline{\mathbf{B}}$ of (28). The sign $\cong$ means: "has the discrete approximation" and $(\,)^*$ denotes the difference expression. Equation (32) holds for a single zero-potential spline $s_0(x, y, z)$. Now a truly remarkable fact appears: *The result (32) is independent of the mesh width h* (gives a constant) and equals the result of the mixed square derivative by the difference operator (31).

Analyzing the results, our basic superposition (28) started from the assumption that the value of the zero-potential density ρ_0 is generated by a superposition of 27 spline functions s_{ijk} summarized in tensor **B**. Why is this superposition identically constant in the "critical region" (sec. 4)?

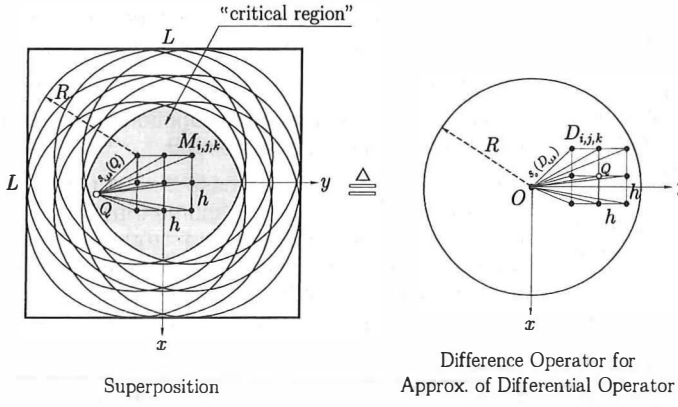

Fig. 8: Superposition ≙ Approximation by Difference Operator

The reason is that our superposition is equivalent to an approximation of our operator by finite differences, cf. Fig. 8. This would not yet be spectacular if the result would essentially depend on h (not only as an inessential scale factor). Since (32) is independent of h, the result of the *difference operator* is identical to the result of the *corresponding differential operator* since we may let $h \to 0$.

Thus the reason why the superposition (28) gives a constant value is the fact that the differential operator (31) naturally gives a constant value (32).

superposition (28) =
= difference operator result (32) =
= differential operator result (31) =
= **const.**

The crucial point in this argument is that (32) is independent of h so that $h \to 0$ does not change anything.

It is proved that the result of a difference operator, which can be written as a three-dimensional tensor, can always be reproduced by superposition in the "critical region".

6. Conclusions

The splines (17) are our basic building blocks for a *practical approximation* of zero-potential densities. Given values at certain data points (e.g. a cubical grid $3 \times 3 \times 3$ points), our spline approximation furnishes the values of a zero-potential density function at any other point in the region under consideration.

The second problem attacked in this paper is of a purely *theoretical significance*. In order to understand a method, it is frequently useful to test it in simple but extreme situations which may be completely unrealistic. (We torture, so to speak, the method in order to force it to show its real character...) Thus we have considered the cubical grid $3 \times 3 \times 3$ and have assumed the data values *equal to 1 at every point of the grid*. This is the "extreme situation" just mentioned and, in fact, it gives the expected unusual behavior which, however, is very instructive: The zero-potential density is identically 1 not only at the data points, but throughout a certain region around them, called "critical region". The result graphically resembles a perfectly smooth plane plateau bounded by wild precipices and jagged mountains.

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